REPRESENTATIONS OF $GL(n,\mathbb{C})$

EXAMPLES STANDARD, SYMMETRIC POWER
EXTERIOR POWER REPS
"ADJOINT REPS".

DEF: A REP'N OF $GL(n,\mathbb{C})$ IS A CONTINUOUS HOMOMORPHISM $\Pi : GL(n,\mathbb{C}) \to GL(V)$ V A F.D. COMPLEX V.S. REQUIRE Π TO BE <u>ANALYTIC</u>.

FOR EXAMPLE IF $V = \mathbb{C}^n$ $GL(V) = GL(n,\mathbb{C})$

THE IDENTITY MAP

$$\Pi_{ST.} : GL(n,\mathbb{C}) \to GL(n,\mathbb{C}) = GL(V)$$

IS A REP'N. BUT COMPLEX CONJUGATION IS NOT:

$$\text{conj} : GL(n,\mathbb{C}) \to GL(n,\mathbb{C})$$
$$A \mapsto \overline{A}$$

NOT ANALYTIC.

GIVEN A REP'N $\pi : \underset{\parallel}{\overset{G}{}}GL(n, \mathbb{C}) \to GL(V)$

ITS CHARACTER $\chi_\pi : G \to \mathbb{C}$

$$\chi_\pi(g) = \operatorname{tr} \pi(g)$$

FACT IS THAT χ_π DETERMINES THE REPRESENTATION.

SKETCH OF PROOF:

$GL(n, \mathbb{C}) \supset U(n)$ UNITARY GROUP

$U(n)$ IS COMPACT. REPS OF $GL(n, \mathbb{C})$ AND $U(n)$ ARE THE SAME IN SENSE THAT EVERY REP'N OF $U(n)$ CAN BE EXTENDED UNIQUELY TO A REP'N OF $GL(n, \mathbb{C})$ AND EVERY REP'N OF $GL(n, \mathbb{C})$ ARISES UNIQUELY THIS WAY.

TO SEE THE REP'NS OF $GL(n, \mathbb{C})$ ARE DETERMINED BY THEIR CHARACTERS WE

PROVE THE SAME THING FOR $U(n)$.

THIS FOLLOWS FROM SCHUR ORTHOGONALITY.

IF π_1, π_2 ARE IRREDUCIBLE

$$\int_{U(n)} \chi_{\pi_1}(g)\, \overline{\chi_{\pi_2}(g)}\, dg = \begin{cases} 1 & \text{IF } \pi_1 \cong \pi_2 \\ 0 & \text{IF } \pi_1 \neq \pi_2 \end{cases}$$

GIVEN A GENERAL REP'N (π, V)

WE CAN DECOMPOSE

$$V = \bigoplus_{i=1}^{N} d_i V_i \qquad \underbrace{\chi_\pi = \sum d_i \chi_i}$$

V_i ARE IRREDUCIBLE. THEN

$$d_i = \int \chi_\pi(g)\, \overline{\chi_{\pi_i}(g)}\, dg$$

SO WE CAN RECONSTRUCT π FROM

ITS CHARACTER. $U(n)$ AND $GL(n, \mathbb{C})$

HAVE INFINITELY MANY IRREP'S, NOT A PROBLEM FOR THIS ARGUMENT.

BETTER: $GL(n,\mathbb{C})$ HAS AN ABELIAN SUBGROUP

$$T = \left\{ \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{pmatrix} \,\middle|\, t_i \in \mathbb{C}^\times \right\} \cong (\mathbb{C}^\times)^n$$

CALLED THE <u>MAXIMAL TORUS</u>.

$$\chi_\pi(g) = \chi_\pi(hgh^{-1}) \text{ FOR ANY}$$

$h \in G = GL(n,\mathbb{C})$ SO THE CHARACTER IS DETERMINED BY ITS VALUES ON A SET OF REPS OF THE CONJUGACY CLASSES. FOR g IN A DENSE SUBSET OF $GL(n,\mathbb{C})$ g IS CONJUGATE

TO A DIAGONAL ELEMENT (JORDAN CANONICAL FORM).

$$g \sim \begin{pmatrix} \boxed{J_1} & & \\ & \boxed{J_2} & \\ & & \ddots \end{pmatrix}$$

ONE BY ONE

J_i IS 1×1 UNLESS g HAS REPEATED EIGENVALS (RARE.)

$$J = \begin{pmatrix} t & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & t \end{pmatrix}$$

SO $\chi_\pi(g)$ IS DET'D BY $\chi_\pi(t)$ WITH $t = \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}$ DIAGONAL.

STANDARD REP'N:

$$GL(n, \mathbb{C}) \longrightarrow GL(n, \mathbb{C})$$

$$\chi_\pi\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} = \sum t_i = s_{(1,0,\dots,0)}(t_1,\dots,t_n)$$

SYMMETRIC k-TH POWER OF STANDARD REP'N

$$\pi_{\vee^k \mathbb{C}^n} : GL(n,\mathbb{C}) \to GL(\vee^k \mathbb{C}^n)$$
$$= GL\left(\binom{n+k-1}{k}, \mathbb{C}\right)$$

$$\chi_\pi\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} = \sum_{i_1 \leq i_2 \leq \cdots \leq i_k} t_{i_1} t_{i_2} \cdots t_{i_k}$$

$$h_k(t_1,\dots,t_n)$$
$$= s_{(k,0,\dots,0)}(t)$$

COMPLETE SYM. POLY.

$$\pi_{\wedge^k \mathbb{C}^n} : GL(n,\mathbb{C}) \to GL(\wedge^k \mathbb{C}^n)$$
$$= GL\left(\binom{n}{k}, \mathbb{C}\right)$$

$$\chi_\pi\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} = \sum_{i_1 < i_2 < \dots < i_k} t_{i_1} \cdots t_{i_k}$$

IT IS ZERO UNLESS $k \leq n$. (ASSUME)

$$= e_k(t_1, \dots, t_n)$$

$$= s_{(\underbrace{1,1,\dots,1}_{k},0,\dots,0)}(t)$$

$$= s_{(1^k)}(t)$$

ADJOINT REP'N:

$$GL(n,\mathbb{C}) \circlearrowleft MAT_n(\mathbb{C}) = \mathbb{C}^{n^2}$$

$$Ad(g)(x) = gxg^{-1}$$

G ACTS ON X

$G \circlearrowleft X$

$g : X \to X$

$$\boxed{\chi_{Ad}\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} = \sum_{i,j} t_i t_j^{-1}}$$

CONSIDER A BASIS E_{ij} OF $MAT_n(\mathbb{C})$

$$E_{ij} = \begin{pmatrix} 0 & \cdots & & \\ 0 & \cdots & 1 & \cdots \\ 0 & \cdots & & \end{pmatrix}$$

1 IN i, j POSITION, ZEROS ELSEWHERE.

$$\pi(t) E_{ij} = \underbrace{t_i t_j^{-1}}_{\text{EIGENVALUES.}} E_{ij}$$

THIS REP'N IS REDUCIBLE:

$I \in \mathrm{MAT}_n(\mathbb{C})$ IS INVARIANT

$$\chi_{\pi}(t) = \sum_{i,j} t_i t_j^{-1}$$

$$= n + \sum_{i \neq j} t_i t_j^{-1}$$

CONTRIB. FROM $i = j$.

INVARIANT

$$\mathrm{MAT}_n(\mathbb{C}) = \underbrace{\mathbb{C} \cdot I}_{1} \oplus \underbrace{\mathrm{MAT}_n^0(\mathbb{C})}_{n^2-1}$$

$\mathrm{MAT}_n^0(\mathbb{C})$ = MATRICES OF TRACE 0.

THE REP'N ON $\mathrm{MAT}_n^0(\mathbb{C})$ OF DIM $n^2 - 1$

IS IRREDUCIBLE. (REDUCED ADJOINT REP'N)

FOR $n = 3$

$$\chi_{Ad}(t) = (n-1) + \sum_{i \neq j} t_i t_j^{-1} \leftarrow$$

$$= (t_1 t_2 t_3)^{-1} \Delta_{(2,1,0)}(t_1, t_2, t_3)$$

CHAR OF DETERMINANT.

FOR $GL(n)$ $\quad \chi_{Ad} = (t_1 \cdots t_n)^{-1} \Delta_{(2,1,\dots,1,0)}(t)$

GIVEN A REP'N (π, V) OF $GL(n, \mathbb{C})$

DECOMPOSE

$$V = \bigoplus_{\mu \in \mathbb{Z}^n} V_\mu \qquad \dim V_\mu \geq 0$$

"WEIGHTS"

$$V_{st} = V_{(1,0,\dots,0)} \oplus V_{(0,1,0,\dots)} \oplus \cdots$$

$$V_\mu = \left\{ v \in V \;\middle|\; \pi\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} v = t^\mu v \right\}$$

$$t^\mu = t_1^{\mu_1} \cdots t_n^{\mu_n}.$$

$$\alpha_i = (0, \dots, \underset{i}{1}, \underset{i+1}{-1}, 0, \dots, 0)$$

"SIMPLE ROOTS".

DEFINE OPERATORS

$$e_i : V_\mu \to V_{\mu + \alpha_i}$$

$$f_i : V_\mu \to V_{\mu - \alpha_i}$$

$$E_{i,i+1} = \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & 1 \\ & & 0 & \ddots \end{pmatrix}$$

$i, i+1$ POSITION.

$$\frac{d}{du} \pi\left(\exp(u E_{i,i+1})\right) v \Big|_{u=0} = e_i(v)$$

FOR GL(3)

(G')

$$e_1(v) = \frac{d}{du}\left(\pi\begin{pmatrix} 1 & u & \\ & 1 & \\ & & 1 \end{pmatrix} v\right)\Bigg|_{u=0}.$$

IT MAY BE CALCULATED THAT

IF $v \in V_\mu$ THEN $e_i(v) = V_{\mu+\alpha_i}$,

SIMILARLY

$$f_i(v) = \frac{d}{du}\pi\left(\exp(u E_{i+1,i})\right) v \Big|_{u=0}$$

FOR Ad OF GL(3):

$$V_{(1,0,-1)} = \mathbb{C} E_{1,2}$$

$$\begin{tikzcd}
 & V_{(0,1,-1)} \arrow[dr, "e_2"] & & V_{(1,0,-1)} \arrow[ll, "e_1"'] \arrow[dr, "e_2"] & \\
V_{(-1,1,0)} \arrow[dr, "e_2"'] & & V_{(0,0,0)} \arrow[ll, "e_1"'] \arrow[dr, "e_2"] & & V_{(1,-1,0)} \arrow[ll, "e_1"'] \\
 & V_{(-1,0,1)} & & V_{(-1,1,0)} \arrow[ll, "e_1"'] &
\end{tikzcd}$$

2-dim'l.

$Ad^0 : GL(3) \circlearrowright Mat_3^0(\mathbb{C})$

IS IT TRUE $f_1\, e_1$ = IDENTITY MAP?

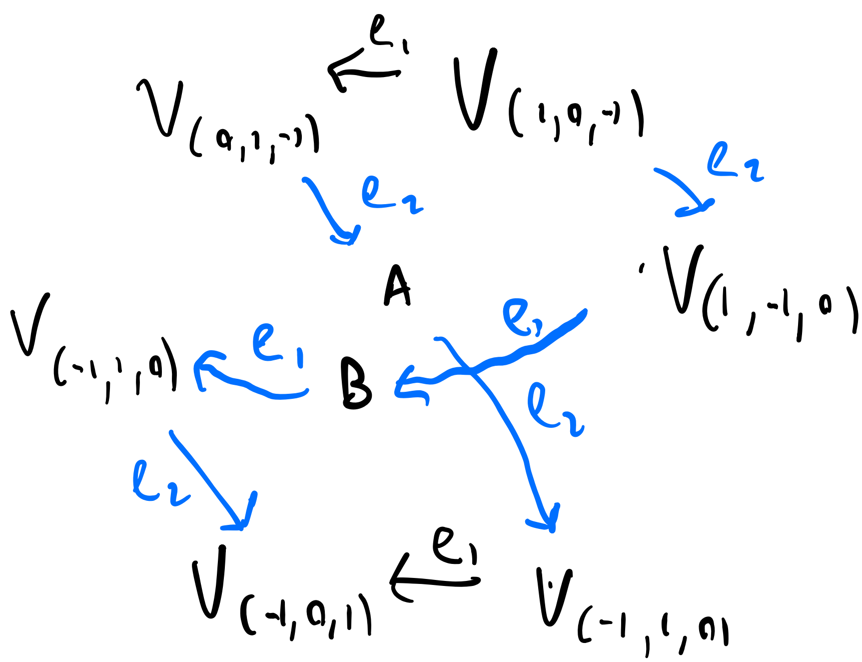
NO THESE MAPS ARE A LITTLE MESSY.

HOWEVER, THIS PICTURE HAS DEFORMATION (FROM QUANTUM GROUPS)

AND CAN BE REPLACED BY A PICTURE IN WHICH THE MAPS ARE

NEAT AND TIDY.

IS THE MODIFIED PICTURE $V_{(0,0,0)}$ WILL DECOMPOSE INTO TWO SPACES AND PICTURE WILL BE:



"CRYSTAL"

$$GL(1) = \mathbb{C}^\times$$

$$U(1) = \mathbb{T} = \{ z \in \mathbb{C} \mid |z| = 1 \}.$$

ANALYTIC REP'N

$$\mathbb{C}^\times \to GL(V)$$

IS DIAGONALIZABLE.

$$a \to \begin{pmatrix} a^{n_1} & & \\ & a^{n_2} & \\ & & \ddots \end{pmatrix}$$

DET'D BY RESTRICTION TO $U(1)$